New dimensions on translations between logics

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Dedicated to the memory of Mário Tourasse Teixeira and Antonio Mário Sette

Abstract. After a brief promenade on the several notions of translations that appear in the literature, we concentrate on three paradigms of translations between logics: *conservative translations*, *transfers* and *contextual translations*. Though independent, such approaches are here compared and assessed against questions about the meaning of a translation and about comparative strength and extensibility of a logic with respect to another.

Keywords. Interpretations, translations, conservative translations, transfers, contextual translations.

1. Interpretations, transformations, translations...

The method of studying inter-relations between logic systems by the analysis of translations between them was originally introduced by Kolmogorov in 1925 (see [29]). The first known 'translations' involving classical logic, intuitionistic logic and modal logic were presented by Kolmogorov, Glivenko, Lewis and Langford, Gödel and Gentzen, most of them developed mainly in order to show the relative consistency of classical logic with respect to intuitionistic logic.

The aim of [29] "is to explain why" the illegitimate use of the principle of excluded middle in the domain of transfinite arguments "has not yet led to contradictions". It introduces the intuitionistic formal logic **B** and the classical propositional calculus **H** and defines inductively a function K associating to every formula α of **H** a formula α^K of **B** by adding a double negation in front of every subformula of α. It is then proven that, given a set of axioms $A = \{\alpha_1, \ldots, \alpha_n\}$, $A \vdash_{\mathbf{H}} \alpha$ implies $A^K \vdash_{\mathbf{B}} \alpha^K$, where $A^K = \{\alpha_1^K, \ldots, \alpha_n^K\}$. Kolmogorov suggests that a similar result can be extended to quantificational systems and, in general, to all known mathematics, anticipating Gödel's and Gentzen's results on the relative consistency of classical arithmetic with respect to intuitionistic arithmetic.

Still related to intuitionism, Glivenko in 1929 proves in [20] that, if α is a theorem of classical propositional logic (**CPL**), the double negation of α is a theorem of intuitionistic propositional logic (**IPL**).

In 1932, during the Mathematical Colloquium held in Vienna, Gödel (apparently not aware of [29]) proved that if α is a theorem of **CPL** then, under a specific translation G, the interpretation $G(\alpha)$ of α is a theorem of Heyting's **IPL**. Gödel shows in [21] that this result is also valid relatively to intuitionistic arithmetic and classical number theory. For him, this result attests that intuitionistic number theory and arithmetic are only apparently weaker than the classical versions, and that the former "contain" the latter. He also introduces in [22] an interpretation i that preserves theoremhood from **IPL** into a system **G**, which is "equivalent" to Lewis' system of strict implication **S4** plus an additional axiom.

The aim of Gentzen in [19] is to show that "the applications of the law of double negation in proofs of classical arithmetic can in many instances be eliminated". He introduces a "transformation" t from the language of **CPL** into **IPL** and proves that $\vdash_{\mathbf{CPL}} \alpha$ if, and only if, $\vdash_{\mathbf{IPL}} t(\alpha)$. As a consequence, a constructive proof of the consistency of classical elementary arithmetic with respect to intuitionistic arithmetic is obtained.

In spite of dealing with inter-relations among specific logic systems, these papers are not interested in the meaning of the concept of translation between logics, several distinct terms having been used by their authors such as interpretation and transformation. Since then, interpretations between logics have been used to different purposes.

Prawitz and Malmnäs survey in [33] these historical papers, and theirs is the first paper in which a general definition for the concept of translation between logic systems is introduced. For them a *translation* from a logic system $\mathbf{S_1}$ into a logic system $\mathbf{S_2}$ is a function t that maps the set of formulas of $\mathbf{S_1}$ into the set of formulas of $\mathbf{S_2}$ such that, for every formula α of $\mathbf{S_1}$,

$$\vdash_{\mathbf{S_1}} \alpha \quad \text{if and only if} \quad \vdash_{\mathbf{S_2}} t(\alpha).$$

The system $\mathbf{S_1}$ is then said to be *interpretable* in $\mathbf{S_2}$ by t. Additionally, $\mathbf{S_1}$ is said to be *interpretable in* $\mathbf{S_2}$ *by* t *with respect to derivability* if, for every set $\Gamma \cup \{\alpha\}$ of formulas in $\mathbf{S_1}$,

$$\Gamma \vdash_{\mathbf{S_1}} \alpha \quad \text{if and only if} \quad t(\Gamma) \vdash_{\mathbf{S_2}} t(\alpha),$$

where $t(\Gamma) = \{t(\beta) \ : \ \beta \in \Gamma\}$. The concept of *schematic translation* is also defined in [33].

Brown and Suszko in [3] construct "a general framework of the theory of abstract logics", concerned with algebraic properties of abstract logics of a same similarity type. Though motivating, they are not interested in the study of inter-relations between general abstract logics (the terms "interpretation" and "translation" are not explicitly mentioned) and "continuous functions" are defined as generalizations of the familiar continuous functions between topological spaces.

Nevertheless, this paper can be considered as an important landmark in the development of the theory of logical translations, anticipating some concepts and results later studied in [8] in a more general setting (see Section 2 below).

Szczerba in [35] defines the concept of interpretation function, that "map structures onto structures". The corresponding functions mapping formulas to formulas are called "translations", but we may say that Szczerba is only concerned with translations between models, and so do not coincide with the approach described in Section 2 below.

Wójcicki in [36] and Epstein in [15] can be considered as the first proposals towards a general systematic study on translations between logics. For Wójcicki, logics are seen as algebras with consequence operators: a logic $(A, \mathcal{C})$ is such that A is a formal language and C is a Tarskian consequence operator in the free algebra of formulas of A. Given two *propositional languages* S_1 and S_2, with the same set of variables, a mapping t from S_1 into S_2 is said to be a *translation* if, and only if:

(i) there is a formula $\varphi(p_0)$ in S_2 in one variable p_0 such that, for each variable p, $t(p) = \varphi(p)$;

(ii) for each connective μ_i in S_1 of arity k there is a formula φ_i in S_2 in the variables $p_1, \ldots, p_k$, such that $t(\mu_i(\alpha_1, \ldots, \alpha_k)) = \varphi_i(t(\alpha_1), \ldots, t(\alpha_k))$ for every $\alpha_1, \ldots, \alpha_k$ in S_1.

A *propositional calculus* is then defined to be a pair $\mathbf{C} = \langle S, C \rangle$, where C is a consequence operator over the language S. Finally, $\mathbf{C}_1 = \langle S_1, C_1 \rangle$ is said to be *translatable* into $\mathbf{C}_2 = \langle S_2, C_2 \rangle$ if there is a mapping t from S_1 into S_2, such that for all $X \subseteq S_1$ and all $\alpha \in S_1$,

$$\alpha \in C_1(X) \quad \text{if and only if} \quad t(\alpha) \in C_2(t(X)).$$

As we shall see, this definition of translation is a particular case of *conservative translation* (cf. Definition 2.5 below).

By its turn, for Epstein [15] a *translation* of a propositional logic $\mathbf{L}$ into a propositional logic $\mathbf{M}$ is thought in semantical terms as a map t from the language of $\mathbf{L}$ into the language of $\mathbf{M}$ such that $\Gamma \models_L \alpha$ if and only if $t(\Gamma) \models_M t(\alpha)$, for every set $\Gamma \cup \{\alpha\}$ of formulas. He also defines the concept of *grammatical translation*.

It can be seen that Kolmogorov's and Gentzen's interpretations are translations in the sense of Prawitz, Wójcicki and Epstein. By its turn, Gödel's ones are translations only in Prawitz' sense (cf. Feitosa and D'Ottaviano [17]).

On the other hand, in [23] (see also [24] and [32]), Goguen and Burstall define a general notion of logic system and his morphisms called *institutions*, within the framework of Category Theory. Institutions generalize Tarski's notion of truth, by considering (abstract) signatures instead of vocabularies, and abstract (categorial) signature morphisms in the place of translations among vocabularies. In this way, the set of sentences are parameterized by abstract signatures. More specifically, it is considered a functor Sen from the category of signatures to the category of sets. Additionally, a (contravariant) functor Mod assigns to every signature Σ its class (category) of models, in such a way that, if $f : \Sigma \longrightarrow \Sigma'$ is a signature morphism then $\mathsf{Mod}(f) : \mathsf{Mod}(\Sigma') \longrightarrow \mathsf{Mod}(\Sigma)$ is a functor between the respective categories of models such that the following condition holds:

$$\mathsf{Mod}(f)(M') \models_\Sigma \varphi \quad \text{iff} \quad M' \models_{\Sigma'} \mathsf{Sen}(f)(\varphi)$$

for every model $M' \in |\mathsf{Mod}(\Sigma')|$ and every sentence $\varphi \in \mathsf{Sen}(\Sigma)$.

2. Running definitions of (conservative) translation

D'Ottaviano in [9] studies variants of Tarskian closure operators characterized by interpretations. Hoppmann in [25][1] uses this characterization and claims that, from a logical point of view, continuous functions between closure structures correspond to functions that preserve deductions, in a remarkable coincidence with the underlying approach of [3]. Given two closure structures $\mathbf{K}_1 = \langle L_1, C_1 \rangle$ and $\mathbf{K}_2 = \langle L_2, C_2 \rangle$, an *interpretation* or a *translation* from $\mathbf{K}_1$ into $\mathbf{K}_2$ is a function $f : \mathcal{P}(L_1) \longrightarrow \mathcal{P}(L_2)$ such that:

(i) for $A, B \subseteq L_1$, $f(C_1(A)) \subseteq C_2(f(A))$;
(ii) the inverse image of a closed set of L_2 is a closed set of L_1;
(iii) if $A \vdash_{C_1} B$ then $f(A) \vdash_{C_2} f(B)$.

This is apparently the first time in the literature the term "translation between general logic systems" is used to mean a function preserving derivability.

Later on, within a research project coordinated by W.A. Carnielli between 1994 and 1997, "Mathematical and computational aspects of translations between logics", sponsored by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP, Brazil), which congregated several Brazilian logicians, philosophers and computer scientists and counted with the participation of foreign colleagues as Michal Krynicki and Xavier Caicedo, new impetus to a wide investigation on translations was found[2]. Da Silva, D'Ottaviano and Sette, motivated by such treatments and explicitly interested in the study of inter-relations between logic systems in general, propose in [8] a general definition for the concept of translation between logics, in order to single out what seems to be in fact the essential feature of a logical translation: logics are characterized as pairs constituted by an arbitrary set (without the usual requirement of dealing with formulas of a formal language) and a consequence operator; translations between logics are then defined as maps preserving consequence relations. In formal terms:

Definition 2.1. A *logic* $\mathbf{A}$ is a pair $\langle A, C \rangle$, where the set A is the *domain* of $\mathbf{A}$ and C is a *consequence operator* in A, that is, $C : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is a function that satisfies, for $X, Y \subseteq A$:

(i) $X \subseteq C(X)$;

[1] [9] and [25] were supervised by Mário Tourasse Teixeira (1925-1993), who suggested, already in the seventies, the concepts and definitions they used.

[2] For the sake of historical consideration, we want to recall here that an unfortunate car accident in one of the project meetings in May 1996 injured several participants and severely impaired A.M. Sette (1939-1998).

(ii) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$;
(iii) $C(C(X)) \subseteq (X)$.

The usual concepts and known results on closure structures are here assumed. We call a theory $X \subseteq A$ *nontrivial in* A if $C_A(X) \neq A$; *trivial in* A otherwise. We say that $a \in A$ *trivializes* A if $C_A(\{a\}) = A$. And $\langle A, C_A\rangle$ is *compact* if there is a finite subset X of A such that $C_A(X) = A$; such an X is called a *trivializing* set.

Of course it is possible to consider logics defined over formal languages:

Definition 2.2. A *logic system* defined over L is a pair $\mathbf{L} = \langle L, C\rangle$, where L is a formal language and C is a structural consequence operator in the free algebra $\mathbf{Form}(L)$ of the formulas of L.

Thus, the only difference between logics and logic systems is that the latter are defined over a formal language.

The proposed general notion of translation between logics (and, in particular, between logic systems) is the following:

Definition 2.3. (cf. [8]) A *translation* from a logic $\mathbf{A}$ into a logic $\mathbf{B}$ is a mapping $t : A \longrightarrow B$ such that $t(C_A(X)) \subseteq C_B(t(X))$ for any $X \subseteq A$.

Clearly, Definition 2.3 can be presented in terms of consequence relations. Thus, if $\mathbf{A}$ and $\mathbf{B}$ are logics with associated consequence relations $\vdash_{C_A}$ and $\vdash_{C_B}$, respectively, then a function $t : A \longrightarrow B$ is a translation if, and only if, for every $\Gamma \cup \{\alpha\} \subseteq \mathbf{Form}(A)$: $\Gamma \vdash_{C_A} \alpha$ implies $t(\Gamma) \vdash_{C_B} t(\alpha)$.

When formal languages are involved, it is useful to consider translations following a well-defined (syntactical) pattern. This motivates the following definition:

Definition 2.4. Let L_1 be a language containing only unary and binary connectives. If L_2 is a language, a translation $t : L_1 \longrightarrow L_2$ is *schematic* if there are formulas $\alpha(p)$, $\beta_\sharp(p)$ of L_2 (for every unary connective $\sharp$ of L_1) depending just on propositional variable p, and formulas $\gamma_\bullet(p, q)$ of L_2 (for every binary connective $\bullet$ of L_1) depending just on propositional variables p, q, such that:
(i) $t(p) = \alpha(p)$, for every atomic formula p of L_1;
(ii) $t(\sharp\varphi) = \beta_\sharp(t(\varphi))$;
(iii) $t(\varphi \bullet \psi) = \gamma_\bullet(t(\varphi), t(\psi))$.

An initial treatment of a theory of translations between logics is presented in [8], where some connections linking translations between logics and uniformly continuous functions between the spaces of their theories are also investigated. An important subclass of translations, the conservative translations, is investigated by D'Ottaviano and Feitosa in [16], [17] and [10].

Definition 2.5. Let $\mathbf{A}$ and $\mathbf{B}$ be logics. A *conservative translation* from $\mathbf{A}$ into $\mathbf{B}$ is a function $t : A \longrightarrow B$ such that, for every set $X \cup \{x\} \subseteq A$,

$$x \in C_A(X) \quad \text{if and only if} \quad t(x) \in C_B(t(X)).$$

The notion of translation in Definition 2.3 accommodates certain maps that seem to be intuitive examples of translations, such as the identity map from intuitionistic into classical logic and the forgetful map from modal logics into classical logic; such cases would be ruled out if the stricter notion of conservative translation were imposed. In this sense, the more abstract notion of translations given in Definition 2.3 is a genuine advance in the scope of relating logic systems, based upon which further unfoldings can be devised.

Note that, in terms of consequence relations, $t : \mathbf{Form}(L_1) \longrightarrow \mathbf{Form}(L_2)$ is a conservative translation when, for every $\Gamma \cup \{\alpha\} \subseteq \mathbf{Form}(L_1)$:

$$\Gamma \vdash_{C_1} \alpha \quad \text{if and only if} \quad t(\Gamma) \vdash_{C_2} t(\alpha).$$

Translations in the sense of Prawitz and Malmnäs do not coincide with conservative translations, nor with translations in the sense of Definition 2.3. Translations in Wójcicki's sense are particular cases of conservative translations, being derivability preserving schematic translations in Prawitz and Malmnäs' sense. Epstein's translations are instances of conservative translations, and his grammatical translations are particular cases of Prawitz and Malmnäs' schematic translations with respect to derivability (and coincide with schematic conservative translations). None of them attempted a more general conception as in Definition 2.3.

Though meaningful extensions of Gödel's well-known interpretability results between **IPL** and **S4** and from **S4** (and **IPL**) into classical arithmetic have been obtained by McKinsey and Tarski, Rasiowa and Sikorski, Solovay, Goldblatt, Boolos and Goodman, Gödel's interpretations do not preserve derivability even in the propositional level, and hence are not translations in the sense of Definition 2.3.

Example 1. The identity function $i : \mathbf{IPL} \longrightarrow \mathbf{CPL}$, both logics considered in the connectives $\neg, \wedge, \vee, \to$, is a translation; but it is not a conservative translation: it suffices to observe that $p \vee \neg p \notin C_{\mathbf{IPL}}(\emptyset)$, while $i(p \vee \neg p) = (p \vee \neg p) \in C_{\mathbf{CPL}}(\emptyset)$. The same occurs with the "forgetful translation" from a modal logic not deriving the formula $\Box p \to p$ into **CPL**.

The next results, taken from [16] and [17] are relevant to the study of general properties of logic systems from the point of view of translations between them.

Theorem 2.6. *(i) A translation* $t : A_1 \longrightarrow A_2$ *is conservative if and only if*

$$t^{-1}(C_2(t(A))) \subseteq C_1(A)$$

for every $A \subseteq A_1$.
(ii) If there is a recursive and conservative translation from a logic system $\mathbf{L}_1$ *into a decidable logic system* $\mathbf{L}_2$*, then* $\mathbf{L}_1$ *is also decidable.*
(iii) Let $\mathbf{L}_1$ *be a logic system with an axiomatics* Λ. *If there is a surjective and conservative translation* $t : \mathbf{L}_1 \longrightarrow \mathbf{L}_2$*, then* $t(\Lambda)$ *is an axiomatics for* $\mathbf{L}_2$. *Additionally, conservative translations preserve non-triviality.*

As an easy consequence, there is no recursive conservative translation from first order logic into **CPL**.

A logic $\mathbf{L}$ *has a deductive implication* if there is a formula $\varphi(p, q)$ depending on two variables such that: $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash \varphi(\alpha, \beta)$. The next result presents conditions for the preservation of Deduction Meta-theorems in the context of deductive implications.

Theorem 2.7. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two logics, $\varphi_1(p, q) \in \mathbf{Form}(L_1)$ and $\varphi_2(p, q) \in \mathbf{Form}(L_2)$. Let $t : \mathbf{L}_1 \longrightarrow \mathbf{L}_2$ be a conservative translation such that $t(\varphi_1(\alpha, \beta)) = \varphi_2(t(\alpha), t(\beta))$. Then: if φ_2 is a deductive implication in $\mathbf{L}_2$ then φ_1 is a deductive implication in $\mathbf{L}_1$; if t is surjective and φ_1 is a deductive implication in $\mathbf{L}_1$ then φ_2 is a deductive implication in $\mathbf{L}_2$.*

A useful method to define conservative translations, as shown in [16] and [17], is the following: given a logic $\mathbf{A}$, consider the equivalence relation

$$x \sim y \quad \text{iff} \quad C(x) = C(y)$$

defined over $\mathbf{A}$, and let $Q : A \longrightarrow A_{/\sim}$ be the quotient map.

Theorem 2.8. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be logics, with the domain of $\mathbf{A}_2$ being denumerable; and let $\mathbf{A}_{1/\sim_1}$ and $\mathbf{A}_{2/\sim_2}$ be the logics co-induced by $\mathbf{A}_1$, Q_1 and $\mathbf{A}_2$, Q_2 respectively.*[3] *Then there is a conservative translation $t : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ if, and only if, there is a conservative translation $t^* : \mathbf{A}_{1/\sim_1} \longrightarrow \mathbf{A}_{2/\sim_2}$. Moreover, if such a t^* exists, then it is injective.*

We observe that the denumerability of A_2 in the hypothesis of the theorem is not necessary if the Axiom of Choice is (explicitly) used in the proof.

Translations into **CPL** seem to be hard to obtain. In particular, [15] proves that, under certain circumstances, such translations do not exist. However, D'Ottaviano and Feitosa present in [12] and [10], respectivelly, non-constructive proofs of the existence of a conservative translation from **IPL** into **CPL**, and from the finite Łukasiewicz' logics into **CPL**.

Conservative translations do not exist in all cases: Scheer in [34], for instance, showed that there is no conservative translation from a cumulative non-monotonic logic into a Tarskian logic, and that there is no surjective conservative translation from a Tarskian logic into a non-monotonic cumulative logic.

D'Ottaviano and Feitosa in [16] and [17] prooved that the category whose objects are topological spaces and whose morphisms are the continuous functions between them is a full sub-category of the bi-complete category of logics and translations; and the category of logics and conservative translations between them is a co-complete subcategory of the category of logics and translations. This is in line with our intuition (shared with [3], [9] and [25]) that topological spaces can be seen as particular cases of logics.

Other developments of the wider notion of translation sprung forth: Carnielli in [4] proposed a new approach to formal semantics for non-classical logics using translations, the so-called *possible-translations semantics*, further investigated by

[3] The logic co-induced by $\mathbf{A}$, Q is $\langle A_{/\sim}, C_{/\sim}\rangle$ such that a set $X \subseteq A_{/\sim}$ is closed iff the set $\{x \in A \ : \ x_{/\sim} \in X\}$ is closed in $\mathbf{A}$.

João Marcos in his Master Dissertation [31]. D'Ottaviano and Feitosa in [11], [12] and [14] introduced several conservative translations involving classical logic, intuitionistic logic, modal logics, paraconsistent logics and many-valued logics. Mauro César Scheer, in his Master Dissertation [34] initiated the study of translations involving cummulative non-monotonic logics. Juliana Bueno-Soler, in her Master Dissertation [2] introduced the *possible-translations algebraic semantics*, in which translations play an essential role for providing an algebraic approach to logics non-algebraizable by traditional methods. Víctor Fernández, in his Ph.D. Thesis [18] used translations in order to investigate combinations of logics, more particularly fibring of logics (cf. Carnielli, Coniglio et al in [5]).

3. Transfers: a model-theoretic dimension to translations

The notion of isomorphism is the indiscernibility principle among algebraic structures, as much as elementary equivalence is the indiscernibility property between logic structures (in the sense that they model the same class of sentences of a given language). The notion of translation between logics seeks to identify the deductive capability and the ability to draw distinctions (the "discriminatory strength", in Humberstone's [27] terminology) of a logic inside another (a non-surjective translation) or onto another (a surjective translation). Now, propositional logics can be advantageously seen as special first-order structures, and in this way we can compare translations with the notions of isomorphism and elementary equivalence. Several properties of logics can be formalized in this way (not all: for instance, compactness is not first-order definable). Also, properties of translations can be imparted and evaluated: so we see the convenience of broader concepts of logic indiscernibility. Such is the aim of the notion of *transfer*.

This section briefly explains the idea of transfers and the underlying model-theoretic approach to translations between logics as developed by Coniglio and Carnielli in [7] (from where all definitions and results in this section are taken).

The basic idea is to consider a formal first-order meta-language to express theoretical (meta)properties of (propositional) logics. Thus the *basic language of abstract logics* is the first-order two-sorted language $\mathbb{L}$ given by

$$\{\mathbf{form}, \mathbf{Sform}\} \cup \{\varepsilon, \vdash\} \cup \{\Cup, \mathbf{s}\} \cup \{\mathbf{0}\}$$

where:

- $\{\mathbf{form}, \mathbf{Sform}\}$ is the set of basic sorts of $\mathbb{L}$;
- ε and $\vdash$ are symbols for predicates of sort $\mathbf{form} \times \mathbf{Sform}$ and $\mathbf{Sform} \times \mathbf{form}$, respectively;
- $\Cup : \mathbf{Sform} \times \mathbf{Sform} \longrightarrow \mathbf{Sform}$ and $\mathbf{s} : \mathbf{form} \longrightarrow \mathbf{Sform}$ are symbols for functions;
- $\mathbf{0}$ is a constant of sort $\mathbf{Sform}$.

Basically, **form** and **Sform** are the sorts for *formulas* and *set of formulas*, respectively; ε and $\vdash$ stand for the membership and consequence relation, respectively; $\Cup$ stands for union of sets; $s(x)$ represents the singleton set $\{x\}$; and $\mathbf{0}$ stands for the empty set. But of course non-standard interpretation for these symbols are possible.

Let $\mathbb{L}'$ be a first-order two-sorted language extending $\mathbb{L}$. A *standard* abstract logic for $\mathbb{L}'$ is a structure $\mathfrak{L}$ for $\mathbb{L}'$ with domains A (for sort **form**) and P (for sort **Sform**) such that $P \subseteq \wp(A) = \{\Gamma \ : \ \Gamma \subseteq A\}$; $\varepsilon_{\mathfrak{L}} \subseteq A \times P$ is the (set-theoretic) membership relation; $\Cup_{\mathfrak{L}} : P \times P \longrightarrow P$ is the (set-theoretic) join operation $\cup$; $\mathbf{s}_{\mathfrak{L}} : A \longrightarrow P$ is given by $\mathbf{s}_{\mathfrak{L}}(a) = \{a\}$ for all $a \in A$; and $\mathbf{0}_{\mathfrak{L}}$ is the empty set $\emptyset$.

¿From Model Theory it is possible to characterize (up to isomorphisms) standard abstract logics by means of a few set-theoretic axioms. It follows that $\mathfrak{L}$ is a standard abstract logic for $\mathbb{L}'$ if and only $P \subseteq \wp(A)$ is a Boolean algebra w.r.t. the set-theoretic operations such that $\emptyset, A \in P$; $\{a\} \in P$ for all $a \in A$; and if $\Gamma \in P$ then $\{a \in A \ : \ \Gamma \vdash_{\mathfrak{L}} a\} \in P$.

It is not hard to see that standard abstract logics are defined along the same lines as Béziau's *Universal Logic* (see, for instance, [1]), that is: a logic is basically a pair formed by a set of entities called formulas and a consequence relation, without assuming any properties. As expected, function symbols of $\mathbb{L}'$ of type $\mathbf{form}^n \longrightarrow \mathbf{form}$ represent n-ary propositional connectives and so a wide class of well-known propositional logics (in particular, propositional Hilbert calculi) can be represented within this framework.

Using the formal meta-language $\mathbb{L}'$ it is possible to express meta-properties of logics such as, for instance, Tarki's axioms and the *Infinite Herz's law*:

$$(\forall Y_1)(\forall Y_2)(\forall Y_3)(\forall x)[(Ent(Y_1,Y_2) \wedge (Y_2 \Cup Y_3 \vdash x)) \ \Rightarrow \ (Y_1 \Cup Y_3 \vdash x)]$$

where $Ent(Y_1,Y_2)$ stands for $(\forall x)[(x \,\varepsilon\, Y_2) \ \Rightarrow \ (Y_1 \vdash x)]$. On the other hand, consistency of a theory can be expressed by the existential formula $(\exists x)\neg(X \vdash x)$.

The generalization of translation between logics is given by the notion of homomorphisms between structures (that is, abstract logics):

Definition 3.1. Let $\mathfrak{L}_i$ be abstract logics over $\mathbb{L}'$ such that $\mathbf{form}_{\mathfrak{L}_i} = A_i$ and $\mathbf{Sform}_{\mathfrak{L}_i} = P_i$ are the universes for sorts **form** and **Sform** in $\mathfrak{L}_i$, respectively (for $i = 1, 2$). A *transfer from* $\mathfrak{L}_1$ *into* $\mathfrak{L}_2$ is an homomorphism $\langle T, T_* \rangle : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$ between structures such that

$$T_*(\Gamma) = T(\Gamma) = \{T(a) \ : \ a \in \Gamma\} \qquad \text{for all } \Gamma \in P_1.$$

Since the mapping $T_* : P_1 \longrightarrow P_2$ is derived from T, it can be omitted. An isomorphic transfer between $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is called an *L-homeomorphism from* $\mathfrak{L}_1$ *to* $\mathfrak{L}_2$. A transfer T is *conservative* if it is an homomorphism such that, for every $(n+m)$-ary predicate symbol R of $\mathbb{L}'$:

$$(\vec{a}; \vec{\Gamma}) \in R_{\mathfrak{L}_1} \quad \text{iff} \quad (T(a_1), \ldots, T(a_n); T(\Gamma_1), \ldots, T(\Gamma_m)) \in R_{\mathfrak{L}_2}$$

for all $\vec{a} = (a_1, \ldots, a_n) \in A_1^n$, $\vec{\Gamma} = (\Gamma_1, \ldots, \Gamma_m) \in P_1^m$. We say that T is an *elementary transfer* if it is an elementary embedding from $\mathfrak{L}_1$ into $\mathfrak{L}_2$.

Clearly, if $T : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$ is a transfer then

$$\Gamma \vdash_{\mathfrak{L}_1} a \text{ implies that } T(\Gamma) \vdash_{\mathfrak{L}_2} T(a),$$

and if T is conservative the following holds:

$$\Gamma \vdash_{\mathfrak{L}_1} a \text{ iff } T(\Gamma) \vdash_{\mathfrak{L}_2} T(a).$$

That is, the usual definitions of translation, conservative translation and L-homeomorphism mentioned in previous sections can be recaptured. It is worth noting that T is an L-homeomorphism iff T is a bijective conservative transfer. L-homeomorphisms determine that the logics involved are *the same* (from the point of view of the language $\mathbb{L}'$). In other words, logics related by L-homeomorphisms are indiscernible by sentences in the language $\mathbb{L}'$.

Proposition 3.2. *Let $T : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$ be a L-homeomorphism. Then for any formula $\Psi(\vec{x}; \vec{Y})$ of $\mathbb{L}'$, $\vec{a} \in A_1^n$ and $\vec{\Gamma}$ in P_1^m:*

$$\mathfrak{L}_1 \models \Psi[\vec{a}; \vec{\Gamma}] \quad \textit{iff} \quad \mathfrak{L}_2 \models \Psi[T(a_1), \ldots, T(a_n); T(\Gamma_1), \ldots, T(\Gamma_m)]$$

where Ψ has at most $x_1, \ldots, x_n$ (of sort **form***) and $Y_1, \ldots, Y_m$ (of sort* **Sform***) as free variables. In particular, for each sentence Ψ in the language $\mathbb{L}$,*

$$\mathfrak{L}_1 \models \Psi \quad \textit{iff} \quad \mathfrak{L}_2 \models \Psi.$$

¿From Model Theory it is known that, in order to obtain a faithful encoding of a structure into another (such as the one of Proposition 3.2), it is enough to have an elementary embedding.

Proposition 3.3. *If T is an elementary transfer then, for any formula $\Psi(\vec{x}; \vec{X})$ of $\mathbb{L}'$ and for any tuples $\vec{a} \in A_1^n$, $\vec{\Gamma} \in P_1^m$:*

$$\mathfrak{L}_1 \models \Psi[\vec{a}; \vec{\Gamma}] \quad \textit{iff} \quad \mathfrak{L}_2 \models \Psi[T(a_1), \ldots, T(a_n); T(\Gamma_1), \ldots, T(\Gamma_m)].$$

For instance, given an elementary transfer $T : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$, if a theory Γ is consistent in $\mathfrak{L}_1$ then the theory $T(\Gamma)$ is consistent in $\mathfrak{L}_2$ (and even in $T(\mathfrak{L}_1)$), because consistency can be expressed through an (existential) formula of the first-order meta-language, as observed above. In fact, it is easy to see that conservative transfers already preserve consistency.

Elementary transfers offer in this way an *intermediate* concept between conservative translation and isomorphism, that is a strict case of conservative translation; this allows to faithfully shift a logic into another, while preserving the meta-properties of the source logic which can be expressed in the formal meta-language $\mathbb{L}'$. As observed in [7], a translation can be considered as "good" whenever it preserves existential formulas (as elementary translations do): the more existential formulas are preserved, the "better" the translation.

4. Contextual translations: still another dimension

The model-theoretic approach typified by transfers as described above is very general but, on the other hand, it requires that every connective of the source logic $\mathcal{L}_1$ (a function symbol of $\mathbb{L}'$) must be translated into another connective. This is a bit restrictive, because often translations between languages are defined in a more liberal way.

Coniglio proposes in [6] a different approach to translations by means of *contextual translations*,[4] which are mappings between languages preserving certain meta-properties of the source logic. As in the case of transfers, these meta-properties are defined in a formal first-order meta-language, acting as a kind of sequent calculus whose rules govern the consequence relation of the logic. This accords with the idea of inferential basis of a logic introduced by J. Łoś and R. Suszko in [30]. In this section we present a simplified version of the definitions and results given in [6].

Consider a set $\mathcal{X} = \{X_i : i \in \mathbb{N}\}$ of *set variables*, a set $\Xi = \{\xi_i : i \in \mathbb{N}\}$ of *schema variables* and a set $\mathcal{V} = \{p_i : i \in \mathbb{N}\}$ of *propositional variables*. A *propositional signature* is a set $C = \{C_i : i \in \mathbb{N}\}$ of sets such that $\mathcal{V} \subseteq C_0$. Elements of C_n are *connectives of arity* n. Let $L(C, \Xi)$ and $L(C)$ be the C-algebra freely generated by $C_0 \cup \Xi$ and C_0, respectively.

An *assertion over* C is a pair $\langle \Upsilon, \varphi \rangle$, written as $\Upsilon \vdash \varphi$, such that Υ is a finite subset of $\mathcal{X} \cup L(C, \Xi)$ and $\varphi \in L(C, \Xi)$. A *meta-property over* C is a pair $\langle \{S_1, \ldots, S_n\}, S \rangle$, written as

$$\frac{S_1 \ \cdots \ S_n}{S}$$

such that S_i (for $i = 1, \ldots, n$) and S are assertions over C.

Consider, for instance, a signature C containing a negation symbol $\neg$ in C_1, a disjunction $\vee$ and a conjunction $\wedge$ in C_2. Then

$$\frac{X_1, \xi_1 \vdash \xi_2 \quad X_1, \neg\xi_1 \vdash \xi_2}{X_1 \vdash \xi_2} \qquad \frac{X_1, \xi_1 \vdash \xi_3 \quad X_1, \xi_2 \vdash \xi_3}{X_1, \xi_1 \vee \xi_2 \vdash \xi_3} \qquad \frac{X_1 \vdash \xi_1 \quad X_2 \vdash \xi_2}{X_1, X_2 \vdash \xi_1 \wedge \xi_2}$$

are meta-properties over C which play the role of logic rules, in which the schema variables ξ_1, ξ_2, ξ_3 act as arbitrary formulas, and the set variables X_1, X_2 act as arbitrary sets of formulas (the context of the rule). This intuition will be formalized below.

A meta-property over C is called *structural* if there is no occurrences of connectives. For instance,

$$\frac{X_1 \vdash \xi_1 \quad X_2, \xi_1 \vdash \xi_2}{X_1, X_2 \vdash \xi_2} \qquad \frac{X_1 \vdash \xi_1}{X_1, X_2 \vdash \xi_1} \qquad \frac{}{X_1 \vdash \xi_1}$$

[4] Called meta-translation in [6].

are structural meta-properties, corresponding to *Cut* rule, *Monotonicity* and *Triviality*. By the very definition, structural meta-properties are defined over any signature.

A *substitution over* C is a map $\sigma : \Xi \to L(C)$. We denote by

$$\hat{\sigma} : L(C, \Xi) \to L(C)$$

the unique homomorphic extension of σ to $L(C)$.

An *instantiation over* C is a map $\pi : \mathcal{X} \to \wp_F(L(C))$, where $\wp_F(L(C))$ denotes the set of finite subsets of $L(C)$.

By instantiating set variables and substituting schema variables, the schematic meta-properties introduced above define concrete instances of meta-properties of logics. Thus, if $\mathcal{L}$ is a propositional logic defined over the language $L(C)$ and (P) is a meta-property over C, we say that $\mathcal{L}$ *has the meta-property* (P) if, for every substitution σ and every instantiation π, $\mathcal{L}$ satisfies the concrete meta-property obtained by applying (σ, π) to (P) (where commas act as unions).

If $\mathcal{L}$ is defined by a sequent calculus with sequents of the form $\Gamma \vdash \varphi$ such that $\Gamma \cup \{\varphi\} \in \wp_F(L(C))$ then the sequent calculus for $\mathcal{L}$ (written in the formal language proposed above) constitutes a basic set of meta-properties of $\mathcal{L}$ which generates every meta-property of $\mathcal{L}$. This is why meta-properties can be seen as formal rules of a formal sequent calulus.

Now we turn our attention to translations. If $f : L(C, \Xi) \longrightarrow L(C', \Xi)$ is a mapping such that $f(\xi) = \xi$ for every $\xi \in \Xi$, and $S = \langle \Upsilon, \varphi \rangle$ is an assertion over C, then $\hat{f}(S)$ is the assertion $\langle \hat{f}[\Upsilon], \hat{f}(\varphi) \rangle$ over C' such that $\hat{f}(\psi) = f(\psi)$ if $\psi \in L(C, \Xi)$, $\hat{f}(X) = X$ if $X \in \mathcal{X}$ and $\hat{f}[\Upsilon] = \{\hat{f}(s) \ : \ s \in \Upsilon\}$. If $(P) = \langle \{S_1, \ldots, S_n\}, S \rangle$ is a meta-property over C then $\hat{f}(P)$ is the meta-property $\langle \{\hat{f}(S_1), \ldots, \hat{f}(S_n)\}, \hat{f}(S) \rangle$ over C'. Note that, if (P) is structural, then $\hat{f}(P)$ coincides with (P).

Definition 4.1. Let $\mathcal{L}$ and $\mathcal{L}'$ be logics defined over signatures C and C', respectively. A *contextual translation* f *from* $\mathcal{L}$ *to* $\mathcal{L}'$, denoted by $f : \mathcal{L} \longrightarrow \mathcal{L}'$, is a mapping $f : L(C, \Xi) \longrightarrow L(C', \Xi)$ such that $\mathcal{L}'$ satisfies the meta-property $\hat{f}(P)$ whenever $\mathcal{L}$ satisfies the meta-property (P). We say that $\mathcal{L}$ is *contextually translatable into* $\mathcal{L}'$ if there exists a contextual translation from $\mathcal{L}$ to $\mathcal{L}'$

Clearly, a contextual translation is a translation in the sense of Definition 2.3. The following results are immediate consequences of the definitions above:

Proposition 4.2. *(a) If* $\mathcal{L}$ *satisfies a structural meta-property which is not satisfied by* $\mathcal{L}'$*, then* $\mathcal{L}$ *is not contextually translatable into* $\mathcal{L}'$*.*
(b) A trivial logic is not contextually translatable into a non-trivial logic.
(c) A monotonic logic is not contextually translatable into a non-monotonic logic.

The last result shows that, in order to be contextually translatable, the logics in question must be compatible in some sense: it is not possible to contextually translate a logic into an arbitrary, unspecified logic. The next examples show that contextual translations and conservative translations are essentially *independent* concepts, and that neither of them entails the other:

Example 2. Let C be the signature for classical propositional logic **CPL** containing $\mathcal{V}$, $\neg$, $\vee$, $\wedge$ and $\rightarrow$, and let C' be the first-order signature obtained from C by substituting each propositional variable $p_i \in \mathcal{V}$ by an unary predicate symbol P_i, and unary connectives $\forall x$ and $\exists x$ for every individual variable x. Let **MON** be the monadic (classical) predicate logic defined over C', and consider a mapping $f : L(C, \Xi) \longrightarrow L(C', \Xi)$ such that

- $f(p_i) = P_i(x)$ and $f(\xi_i) = \xi_i$ (for $i \in \mathbb{N}$);
- $f(\varphi \# \psi) = f(\varphi) \# f(\psi)$ for $\# \in \{\vee, \wedge, \rightarrow\}$;
- $f(\neg\varphi) = \neg f(\varphi)$.

Then f : **CPL**$\longrightarrow$**MON** is a conservative translation, but it is not a contextual translation from **CPL** to **MON**. In fact, let (P) be the following meta-property of **CPL** (the Deduction Meta-theorem):

$$\frac{X_1, \xi_1 \vdash \xi_2}{X_1 \vdash \xi_1 \rightarrow \xi_2}$$

Then $\hat{f}(P)$ is (P), which is not valid in **MON**: take, for instance, the substitution σ and the instantiation π over C' such that $\sigma(\xi_1) = P_1(x)$, $\sigma(\xi_2) = \forall x P_1(x)$ and $\pi(X_1) = \emptyset$. Thus $P_1(x) \vdash_{\mathbf{MON}} \forall x P_1(x)$ holds (by *Generalization* rule), but $\vdash_{\mathbf{MON}} P_1(x) \rightarrow \forall x P_1(x)$ is not true. That is, **MON** does not satisfy $\hat{f}(P)$.

Example 3. Let C be as above, and let C' be the signature extending C by adding a unary connective $\Box$ (the modal "necessity" operator). Consider normal modal logic **S4**, in which the consequence relation is defined from theoremhood as usual: $\Gamma \vdash \varphi$ iff there exists $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $\vdash (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \varphi$ (for $\Gamma \neq \emptyset$). This forces **S4** to satisfy the Deduction Meta-theorem. Let $f : L(C, \Xi) \longrightarrow L(C', \Xi)$ be a mapping such that

- $f(p) = \Box p$ (if $p \in \mathcal{V}$);
- $f(\xi) = \xi$ (if $\xi \in \Xi$);
- $f(\varphi \# \psi) = f(\varphi) \# f(\psi)$ for $\# \in \{\wedge, \vee\}$;
- $f(\neg\varphi) = \Box\neg f(\varphi)$;
- $f(\varphi \rightarrow \psi) = \Box(f(\varphi) \rightarrow f(\psi))$.

Then f : **INT**$\longrightarrow$**S4** is a conservative translation (cf. [15]). On the other hand, f is not a contextual translation: the translation of the Deduction Meta-theorem (P) is the meta-property

$$\frac{X_1, \xi_1 \vdash \xi_2}{X_1 \vdash \Box(\xi_1 \rightarrow \xi_2)}$$

which is not valid in **S4**: take $\sigma(\xi_1) = p_1$, $\sigma(\xi_2) = p_2$ and $\pi(X_1) = \{p_1 \rightarrow p_2\}$.

Example 4. Let C be as above, and let **CPL** and **IPL** be classical and intuitionistic propositional logic defined over C, respectively. Then the identity map $i : L(C, \Xi) \longrightarrow L(C, \Xi)$ is a contextual translation i : **IPL**$\longrightarrow$**CPL**, since every sequent rule valid in **IPL** is also valid in **CPL**. On the other hand, i is not a conservative translation from **IPL** to **CPL**, as observed in Example 1.

Example 5. Let C be as above, and let C' be a non-empty subsignature of C, for instance containing just $\mathcal{V}$, $\wedge$ and $\vee$. Let $\mathcal{L}'$ be the fragment of **CPL** defined over C', and let $i : L(C', \Xi) \longrightarrow L(C, \Xi)$ be the inclusion map. Then $i : \mathcal{L}' \longrightarrow \mathbf{CPL}$ is both, a conservative and a contextual translation.

Example 6. Let C be as above, and let C' be the signature just containing $\mathcal{V}$, $\neg$ and $\rightarrow$. Let $\mathcal{L}'$ be the fragment of **CPL** defined over C', and let $f : L(C, \Xi) \longrightarrow L(C', \Xi)$ be a mapping such that

- $f(s) = s$ for $s \in \mathcal{V} \cup \Xi$;
- $f(\varphi \wedge \psi) = \neg(f(\varphi) \rightarrow \neg f(\psi))$;
- $f(\varphi \vee \psi) = \neg f(\varphi) \rightarrow f(\psi)$;
- $f(\varphi \rightarrow \psi) = f(\varphi) \rightarrow f(\psi)$;
- $f(\neg\varphi) = \neg f(\varphi)$.

Then $f : \mathbf{CPL} \longrightarrow \mathcal{L}'$ is both, a conservative and a contextual translation. The same example applies if C' just contains $\mathcal{V}$, $\neg$ and $\vee$, or $\mathcal{V}$, $\neg$ and $\wedge$.

Examples 2 and 3 above witness cases of conservative but non-contextual translations, while Example 4 is a contextual but non-conservative translation. On the other hand, Examples 5 and 6 are paradigmatic cases of translations in both senses (conservative and contextual): the former concerns embeddings of fragments of **CPL** to itself, while the latter concerns rewrittings of **CPL** into truth-functionally complete fragments of itself.

Contextual translations between logics refine the concept of mere translations between logics as much as linguistic contextual translations are preferable over simple-minded literal linguistic translations. This refined concept helps us to analyze not only the complicate question of "How a logic can be translated into another one?", but also a triad of provoking questions:

- What is the essential meaning of a translation?
- How can a logic be extended?
- When can a logic be seen as a legitimate sublogic of another one?

Towards the first question, it is remarkable that source logics translated by conservative translations are similar (in a precise way) to their images. But the image of a logic under a conservative translation could very well be a small, perhaps banal, fragment of the target logic. Consider, for instance, the following example adapted from [7]: let $\mathcal{L}$ and $\mathcal{L}'$ be Tarskian logics defined over C such that $L(C) = \{\psi_i \ : \ i \in \mathbb{N}\}$ is denumerable (where $\psi_i \neq \psi_j$ if $i \neq j$). Suppose also that $\mathcal{L}$ is a trivial logic and that $\mathcal{L}'$ has a denumerable set $T = \{\varphi_i \ : \ i \in \mathbb{N}\}$ of theorems (where $\varphi_i \neq \varphi_j$ if $i \neq j$). Let $f : L(C) \longrightarrow L(C)$ such that $f(\psi_i) = \varphi_i$ for every $i \in \mathbb{N}$. Then f is an injective conservative translation from $\mathcal{L}$ to $\mathcal{L}'$. The image of $\mathcal{L}$ is the restriction of $\mathcal{L}'$ to T which, in fact, is the trivial logic: everything follows from everything as much as theorems are concerned. This is how a trivial logic is injectively translated into a (possibly) non-trivial logic: the image (injective copy) of $\mathcal{L}$ within $\mathcal{L}'$ is banal with respect to the whole logic $\mathcal{L}'$. Does this translation contribute with some essential information about the target logic? None, not even that $\mathcal{L}'$ has a trivial sublogic, as $\mathcal{L}'$ can be any logic (for instance, a logic without trivializing particle $\perp$).

This kind of phenomenon is inherent in the nature of (conservative) translations, which allow a logic to be translated within a proper fragment of another. Contextual translations, on the other hand, require that a logic be translated onto the full target logic, in the sense that meta-rules of the former are globally valid in the latter.

A translation tradeoff can be identified here: translations, intended in the sense of Definition 2.3, help to clarify the maxim expressed in [26] and [27] according to which the weaker a logic is (in its deductive strength) the stronger it is in its ability to draw distinctions (in its discriminatory strength). So the weaker a logic is, the more logics could be translated into it. This point of view is also supported by Janssen in [28]

But of course, in the opposite direction, conservative contextual translations are much stronger: examples are hard to find, but widely expressive.

In the direction of the second and third questions above, contextual translations also pave the way for understanding how a logic can be strengthened: the schema variables above codify the "logic space" for such expansion.

Also, we may conjecture that a contextual translation from $\mathcal{L}$ to $\mathcal{L}'$ is only possible when the discriminatory strength of $\mathcal{L}$ overrides that of $\mathcal{L}'$: all the examples suggest that $\mathcal{L}$ is a sublogic of $\mathcal{L}'$ or $\mathcal{L}'$ extends $\mathcal{L}$ while preserving derivability and meta-properties.

The above examples show, for instance, that **IPL** is a "good" sublogic of **CPL**, in the sense that every meta-property of (or rather, a property in the context of) **IPL** is enjoyed by **CPL**. On the other hand, **CPL** is not "such a good" sublogic of **MON** since the latter does not enjoy the Deduction Meta-theorem (cf. Examples 14(1) and 14(3) above). This may be converted into a very reasonable criterion to consider a logic as a legitimate sublogic of another one.

At the same time that our account of contextual translations helps to explicate the meaning of expanding and translating logics, it also opens space for problems: as two immediate examples, can the problems of admissible and structural rules in logics be explained or rephrased in terms of translations? Can the diverse meanings of completeness (Post, Halldén) be elaborated with the help of translations?

Being able to face problems like those, rooted on more than three decades of research among Brazilian logicians, is the best our account on translations would aspire.

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